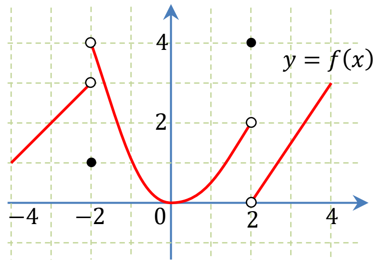
Problem Set 2

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 2 covers materials from §2.1 – §2.3.

1. A portion of the graph of a function $f: \mathbb{R} \to \mathbb{R}$ is shown below.



   (a) Compute $f(-2)$, $\lim_{x\to -2^-} f(x)$, $\lim_{x\to -2^+} f(x)$, $f(2)$, $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

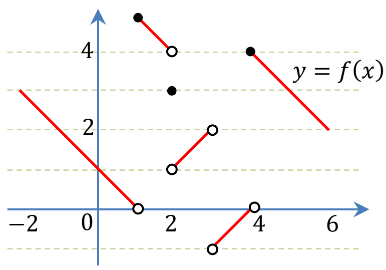
   (b) Now let $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ be the functions defined respectively by
   $$g(x) = f(|x-2|) \qquad \text{and} \qquad h(x) = f(|x|-2).$$

   (i) Compute $g(0)$, $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$.

   (ii) Compute $h(0)$, $\lim_{x\to 0^-} h(x)$ and $\lim_{x\to 0^+} h(x)$.

   (iii) For each of the following limits
   $$\lim_{x\to 0} f(x-2), \qquad \lim_{x\to 0} g(x), \qquad \lim_{x\to 0} h(x),$$
   explain briefly whether the limit exists or not.

2. The graph of a function f is shown below.



   Using information from this graph, compute the limit
   $$\lim_{x\to 2} f\big(3 - f(4-x)\big)$$
   if it exists.

3. Let f be a function such that

$$\lim_{x\to 0}\frac{\ln(1+x)}{f(x)}=1.$$

Evaluate the limit $\lim_{x\to 0} f(x)$.

4. Suppose that the limit

$$\lim_{x\to -1}\frac{x^3-ax^2-x+4}{x+1}$$

exists as a finite real number. Find the value of a and thus evaluate the limit.

5. Let

$$f(x)=\frac{|x^2+x-2|}{x+2}.$$

Evaluate each of the limits $\lim_{x\to -2^-} f(x)$, $\lim_{x\to -2^+} f(x)$ and $\lim_{x\to -2} f(x)$ if it exists.

6. Let

$$f(x)=\begin{cases} x^2+1 & \text{if } x<-1 \\ \sqrt{x+1} & \text{if } -1\le x<1 \\ 2\sin\dfrac{\pi x}{4} & \text{if } x\ge 1 \end{cases}.$$

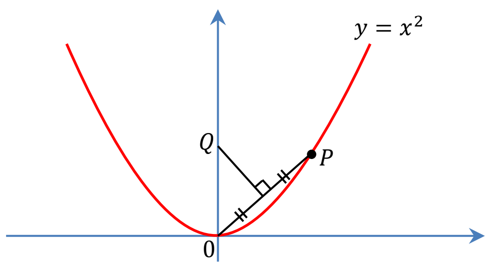
Compute $\lim_{x\to a} f(x)$ for each real number a.

7. Let

$$f(x)=\begin{cases} x^3+a & \text{if } x<0 \\ a\sin\dfrac{\pi x}{3}+b & \text{if } 0\le x<2 \\ 3 & \text{if } x=2 \\ \log_2 x^{b+1} & \text{if } x>2 \end{cases}.$$

If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 2} f(x)$ both exist, find the values of a and b.

8. The following diagram shows a point P on the parabola $y=x^2$ and the point Q at which the perpendicular bisector of OP intersects the y-axis.



As P approaches the origin, is there a limiting position of the point Q? What is this limiting position?

9. Evaluate each of the following limits, if it exists:

(a) $\lim_{x\to 1}\frac{x^6-1}{x-1}$

(b) $\lim_{x\to -1}\frac{x^{1013}+1}{x+1}$

(c) $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

(d) $\lim_{x\to 4}\frac{3x\sqrt{x+5}-12\sqrt{x+5}}{3-\sqrt{x+5}}$

(e) $\lim_{x\to 1}(x-1)\cot \pi x$

(f) $\lim_{x\to 0}\sqrt{x^2+x^3}e^{\sin\frac{\pi}{x}}$

(g) $\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}$

(h) $\lim_{x\to 0}\frac{\arctan 2x}{\sin x}$

10. Evaluate each of the following limits, if it exists:

(a) $\lim_{x\to +\infty}\frac{\sin^2 x}{x^2}$

(b) $\lim_{x\to +\infty}\frac{x+\cos x}{x-1}$

(c) $\lim_{x\to +\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$

(d) $\lim_{x\to -\infty}\left(\sqrt{9x^2+3x+1}+3x\right)$

(e) $\lim_{x\to +\infty}\frac{2xe^x\sin\frac{1}{x}+e^x\cos\frac{1}{x}}{e^x+e^{-x}}$

(f) $\lim_{x\to +\infty}\arctan(x-x^6)$

11. A large tank initially contains 5000 L of pure water. Brine (salt solution) that contains 30 g of salt per liter is then pumped into the tank at a rate of 25 L per minute.

(a) What is the concentration of salt in the tank (in g/L) after t minutes?

(b) As time passes (i.e. as $t\to +\infty$), what happens to this concentration of salt?

12. Consider an object falling under gravity and is experiencing air resistance. It can be shown that the velocity $v(t)$ of the object at time t is given by

$$v(t) = ue^{-\frac{\gamma}{m}t}+\frac{mg}{\gamma}\left(1-e^{-\frac{\gamma}{m}t}\right),$$

where $m>0$ is the (constant) mass of the object, $\gamma>0$ is a constant called the "coefficient of air resistance", g is the constant acceleration due to gravity, and u is the initial velocity of the object.

(a) Show that as time passes, the velocity $v(t)$ gets closer and closer to a constant value, i.e. show that the limit

$$\lim_{t\to +\infty} v(t)$$

exists as a number.

(b) The limit in (a) is called the ***terminal velocity*** of the falling object. How does this limit depend on the initial velocity u of the object?

13. Let

$$f(x) = \frac{2}{x^2 - 1} \qquad \text{and} \qquad g(x) = \frac{1}{x - 1}.$$

Evaluate each of the following one-sided limits.

(a) $\lim_{x \to 1^-} \big(f(x) + g(x)\big)$

(b) $\lim_{x \to 1^+} \big(f(x) + g(x)\big)$

(c) $\lim_{x \to 1^-} \big(f(x) - g(x)\big)$

(d) $\lim_{x \to 1^+} \big(f(x) - g(x)\big)$

14. Let f be a function and suppose that

$$\lim_{x \to 0} \frac{f(x)}{x} = 5.$$

Evaluate each of the following limits, if it exists.

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 0} \frac{f(x)}{x^2}$

(c) $\lim_{x \to 0} \frac{f(x)}{x^3}$

15. Find all the asymptotes of the graph of each of the following functions.

(a) $f(x) = \frac{3x^5 + 3x^3}{2x^4 - 2}$

(b) $f(x) = \frac{2x + e^x}{x - 3}$

16. Let m and c be real numbers and let $f: [0, +\infty) \to \mathbb{R}$ be a function. If

$$\lim_{x \to +\infty} [f(x) - (mx + c)] = 0,$$

show that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = m \qquad \text{and} \qquad \lim_{x \to +\infty} (f(x) - mx) = c.$$